Differentialgeometrie auf Mannigfaltigkeiten
FSU Jena - WS 07/08
Serie 02 - Lösungen

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17. April 2009

Aufgabe 04

Sei M eine C^∞ Mannigfaltigkeit und $P \in M$. Seien in M_P die Skalarmultiplikation und Vektor-Addition als punktweise Addition erklärt.
Um zu zeigen dass M_P ein linearer Raum ist, bestätigen wir folgende erforderten Eigenschaften:
Für $\lambda, \mu, \kappa \in \mathbb{R},\ x, y \in M_P$ und $f, g \in \mathcal{F}(P)$ gilt

a) Assoziativität:
$$\lambda \cdot (\mu \cdot x) = (\lambda \cdot \mu) \cdot x$$

b) Distributivität:
$$\lambda \cdot (x + y)f = \lambda \cdot (xf + yf) = \lambda \cdot xf + \lambda yf = (\lambda x + \lambda y)f \ \Rightarrow\ \lambda \cdot (x + y) = \lambda x + \lambda y$$
$$(\lambda + \mu)(xf) = \lambda xf + \lambda xf = (\lambda x + \lambda\mu)f \ \Rightarrow\ (\lambda + \mu)x = \lambda x + \lambda\mu$$

c) Neutralität:
$$1 \cdot x = x$$

d) Abgeschlossenheit bzgl. der Vektoraddition bzw. Skalarmultiplikation:
Betrachten $z := \kappa x + y$ im Sinne von $zf := \kappa \cdot xf + yf$. Wir wollen zeigen dass $z \in M_P$.
$$z(\lambda f + \mu g) = \kappa \cdot x(\lambda f + \mu g) + y(\lambda f + \mu g) = \lambda(\kappa \cdot xf + yf) + \mu(\kappa \cdot xg + yg) = \lambda zf + \mu zg$$
$$z(f \cdot g) = (\kappa x + y)(fg) = \kappa \cdot x(fg) + y(fg) = (\kappa \cdot xf) \cdot g(P) + f(P) \cdot (\kappa \cdot xg) + (yf)g(P) + f(P)(yg)$$
$$= (\kappa \cdot xf + yf)g(P) + (\kappa \cdot xg + yg)f(P) = (zf)(P) + f(P)(zg)$$

e) Als Nullelement von M_P wird die konstante Funktion 0 gewählt. $\square$

Aufgabe 05

Definieren: $X_P := X(P),\ Y_P := Y(P),\ f_P := f(P)$
Bemerkung: Es gilt

$$X(\lambda f + \mu g) = \lambda X f + \mu X g \ \wedge \ X(fg) = (Xf)g + f(Xg)$$

da

$$X_P(\lambda f + \mu g) = \lambda X_P f + \mu X_P g \ \wedge \ X_P(fg) = (X_p f)g(P) + f(P)(X_p g)$$

gilt. Sei $Z : \mathcal{F}(P) \to \mathbb{R}$ definiert durch

$$Zf := XYf(P) - YXf(P) \equiv X_P Y f - Y_P X f$$

Wir wollen zeigen dass $Z \in M_P$.
Linearität: Seien $\lambda, \mu \in \mathbb{R},\ f, g \in \mathcal{F}(M)$. Dann gilt

$$z(\lambda f + \mu g) = XY(\lambda f + \mu g) - YX(\lambda f + \mu g) = X(\lambda Y f + \mu Y g) - Y(\lambda X f + \mu X g)$$

$$= \lambda XYf + \mu XYg - \lambda YXf - \mu YXg$$

$$= \lambda \cdot (XYf - YXf) + \mu \cdot (XYg - YXg) = \lambda Zf + \mu Zg$$

Produktregel:

$$Z(fg) = XY(fg) - YX(fg) = X[(Yf)g + f(Yg)] - Y[(Xf)g + f(Xg)]$$

$$= (XYf)g + (Xg)(Yf) + (Xf)(Yg) + (XYg)f - (YXf)g - (Yg)(Xf) - (Yf)(Xg) - (YXg)f$$

$$= (XYf)g + (XYg)f - (YXf)g - (YXg)f = [(XY - YX)f]\,g + [(XY - YX)g]\,f = (Zf) \cdot g + f \cdot (Zg)$$

Somit ist $Z \in M_P$.

Aufgabe 06

Seien $f, g, h \in \mathcal{F}(M)$ und $X, Y \in \mathcal{X}(M)$.

$$[fX, gY](P)h = (fX)_P(gY)h - (gY)_P(fX)h = f_P X_P\,(g(Yh)) - g_P Y_P\,(f(Xh))$$

$$= f_P \cdot ((X_p Y h) g_P + (Yh)_P (X_p g)) - g_P \cdot ((Y_P X h) f_P + (Xh)_P (Y_P f))$$

$$= f_P \cdot g_P \cdot (X_P Y h - Y_P X h) + \underbrace{f_P(X_P g)}_{f_P(Xg)_P} \cdot (Yh)_P - \underbrace{g_P(Y_P f)}_{g_P(Yf)_P} \cdot (Xh)_P$$

$$\Rightarrow \ [fX, gY] = fg[X, Y] + f(Xg) \cdot Y - g(Yf) \cdot X$$

Aufgabe 07

$$\left\{\sum_j \left[\sum_i \left(x^i\partial_i Y^j - Y^i\partial_i X^j\right)\right]\partial_j\right\}(P)f = \sum_j \left[\sum_i \left(X^i\partial_i Y^j - Y^i\partial_i X^j\right)(P)\right]\cdot \partial_j(P)f$$

$$= \sum_j \left[\sum_i \left(X_P^i(\partial_i Y^j)_P - Y_P^i(\partial_i X^j)_P\right)\right]\cdot(\partial_j f)_P$$

$$= \sum_j\sum_i X_P^i\cdot(\partial_i(P)Y^j)\cdot(\partial_j f)_P - \sum_i\sum_j Y_P^j\cdot(\partial_j(P)X^i)\cdot(\partial_i f)_P$$

$$= \sum_i X_P^i\cdot\left[\sum_j(\partial_i(P)Y^j)\cdot(\partial_j f)_P\right] - \sum_j Y_P^j\cdot\left[\sum_i(\partial_j(P)X^i)\cdot(\partial_i f)_P\right]$$

$$\overset{a}{=} \sum_i X_P^i\cdot\left\{\partial_i(P)\sum_j Y^j\cdot(\partial_j f) - \sum_j Y_P^j(\partial_i(P)\partial_j f)\right\} - \sum_j Y_P^j\cdot\left\{\partial_j(P)\sum_i X^i(\partial_i f) - \sum_i X_P^i(\partial_j(P)\partial_i f)\right\}$$

$$= \sum_i\left[X_P^i\partial_i(P)\sum_j Y^j\cdot(\partial_j f)\right] - \sum_j\left[Y_P^j\partial_j(P)\sum_i X^i(\partial_i f)\right] - \sum_i\sum_j X_P^i Y_P^j(\partial_i\partial_j f)_P + \underbrace{\sum_j\sum_i Y_P^j X_P^i(\partial_j\partial_i f)_P}_{\sum_i\sum_j Y_P^j X_P^i(\partial_i\partial_j f)_P}$$

$$= \left(\sum_i X^i\partial_i\right)_P\left(\sum_j Y^j\partial_j f\right) - \left(\sum_j Y^j\partial_j\right)_P\left(\sum_i X^i\partial_i f\right) = \left[\sum_{i=1}^n X^i\partial_i,\ \sum_{j=1}^n Y^j\partial_j\right](P)f = [X,Y](p)f \quad \square$$

(a) : Produktregel bei Tangentenvektoren.

Variante:

$$[X,Y] = \left[\sum_i X^i\partial_i, \sum_k X^k\partial_k\right] = \sum_{ik}\left[X^i\partial_i, X^k\partial_k\right] = \sum_{ik} X^i X^k\underbrace{[\partial_i,\partial_k]}_{0} + \sum_{ik} X^i\left(\partial_i Y^k\right)\partial_k - \sum_{ik} Y^k\left(\partial_k X^i\right)\partial_i$$

$$= \sum_j\left[\sum_i X^i\left(\partial_i Y^j\right)\partial_j - \sum_k Y^k\left(\partial_k X^j\right)\partial_j\right] = \sum_j\left[\sum_i X^i\partial_i Y^j - \sum_i Y^i\partial_i X^j\right]\partial_j \quad \square$$

Aufgabe 08

$$\{[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]\}\,(P)f$$

$$=[X,Y]_P\,(Zf)-Z_P\,([X,Y]\,f)+[Y,Z]_P\,(Xf)-X_P\,([Y,Z]\,f)+[Z,X]_P\,(Yf)-Y_P\,([Z,X]\,f)$$

$$=[X_P(Y(Zf))-Y_P(X(Zf))]-[Z_P(X(Yf))-Z_P(Y(Xf))]$$

$$+[Y_P(Z(Xf))-Z_P(Y(Xf))]-[X_P(Y(Zf))-X_P(Z(Yf))]$$

$$+[(Z_P(X(Yf))-X_P(Z(Yf))]-[Y_P(Z(Xf))-Y_P(X(Zf))]$$

$$=0\quad\Box$$