Analysis III
FSU Jena - WS 07/08
Serie 08 - Lösungen

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Aufgabe 00

$$\frac{\partial U}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial u}{\partial x}\cos\varphi + \frac{\partial u}{\partial y}\sin\varphi$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}x + \frac{\partial u}{\partial y}y\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}x + \frac{\partial u}{\partial y}y\right)\frac{\partial y}{\partial r}$$

$$= \left(\frac{\partial^2 u}{\partial x^2}x + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y}\right)\cos\varphi + \left(\frac{\partial^2 u}{\partial y\partial x}x + \frac{\partial^2 u}{\partial y^2}y + \frac{\partial u}{\partial y}\right)\sin\varphi$$

$$\frac{\partial U}{\partial \varphi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \varphi} = -\frac{\partial u}{\partial x}\underbrace{r\sin\varphi}_{y} + \frac{\partial u}{\partial y}\underbrace{r\cos\varphi}_{x}$$

$$\frac{\partial^2 U}{\partial \varphi^2} = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial x}y + \frac{\partial u}{\partial y}x\right)\frac{\partial x}{\partial \varphi} + \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial x}y + \frac{\partial u}{\partial y}x\right)\frac{\partial y}{\partial \varphi}$$

$$= \left(-\frac{\partial^2 u}{\partial x^2}y + \frac{\partial^2 u}{\partial x\partial y}x + \frac{\partial u}{\partial y}\right)(-r\sin\varphi) + \left(-\frac{\partial^2 u}{\partial y\partial x}y - \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2}x\right)r\cos\varphi$$

$$\frac{\partial^2 U}{\partial z^2} = \frac{\partial^2 u}{\partial z^2}$$

$$\to\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2}\left(\cos^2\varphi + \sin^2\varphi\right) + \frac{\partial^2 u}{\partial y^2}\left(\sin^2\varphi + \cos^2\varphi\right) + \frac{\partial^2 u}{\partial x\partial y}\left(2\cos\varphi\sin\varphi - 2\cos\varphi\sin\varphi\right)$$

$$+ \frac{\partial u}{\partial x}\left(-\frac{\cos\varphi}{r} + \frac{\cos\varphi}{r}\right) + \frac{\partial u}{\partial y}\left(-\frac{\sin\varphi}{r} + \frac{\sin\varphi}{r}\right) + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad \square$$

Aufgabe 01

Sei

$$u(x,y) = a_{00} + a_{10}x + a_{20}x^2 + a_{01}y + a_{02}y^2 + a_{11}xy$$

ein harmonisches Polynom n-ten Grades, $n \leq 2$, das heisst

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2a_{20} + 2a_{02} = 0$$

Dann kann a_{ij} beliebig sein, mit der einzigen Einschränkung dass

$$a_{20} \stackrel{!}{=} -a_{02}$$

Aufgabe 02

Sei

$$U : \mathbb{R}^2 \to \mathbb{R}, \ U(\vec{r}) = U(r)$$

eine harmonische radialsymmetrische Funktion, das heisst

$$\Delta U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) = 0$$

Dann folgt

$$\frac{d}{dr}\left(r\frac{dU}{dr}\right) = 0 \ \to \ \frac{dU}{dr} = \frac{C}{r} \ \to \ U = C\ln r + \mathcal{A}$$

mit $C, \mathcal{A} \in \mathbb{R}$

Aufgabe 03

Sei $u = u(x,y) : \mathbb{R}^2 \to \mathbb{R}$ mit

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Wir machen den Ansatz $u(x,y) = f(x)\cdot g(y)$ und gehen damit in die Differentialgleichung ein:

$$0 = \Delta u = g\cdot\frac{\partial^2 f}{\partial x^2} + f\cdot\frac{\partial^2 g}{\partial y^2} = g\cdot\frac{d^2 f}{dx^2} + f\cdot\frac{d^2 g}{dy^2}$$

$$\to \ \frac{1}{f}\cdot\frac{d^2 f}{dx^2} = -\frac{1}{g}\cdot\frac{d^2 g}{dy^2} \quad \forall\, x,y$$

Aus oberer Gleichung und deren Invarianz bzgl. (x,y) folgt

$$\frac{1}{f}\cdot\frac{d^2 f}{dx^2} = -\frac{1}{g}\cdot\frac{d^2 g}{dy^2} =: h : const$$

Wir bekommen also 2 Gewöhnliche DGL, deren Lösungen jeweils lauten:

$$f_h(x) = A_h e^{x\sqrt{h}} + B_h e^{-x\sqrt{h}} \ \ : h \neq 0$$

$$g_h(y) = C_h e^{y\sqrt{-h}} + D_h e^{-y\sqrt{-h}} \ \ : h \neq 0$$

$$f_0(x) = A_0 + B_0 x, \quad g_0(y) = C_0 + D_0 y$$

Sind $u_h(x,y) = f_h(x)\cdot g_h(y)$ und $u_k = f_k(x)\cdot g_k(y)$ Lösungen der DGL, so ist auch

$$u_{hk} := \lambda\cdot u_h(x,y) + \mu\cdot u_k(x,y), \quad \lambda,\mu \in \mathbb{R}$$

eine Lösung der DGL. Somit ergibt sich die allgemeine Lösung als

$$\boxed{u(x,y) = (A(0) + B(0)x)\cdot(C(0) + D(0)y) + \int\limits_{\mathbb{R}\setminus\{0\}} \left[A(h)e^{x\sqrt{h}} + B(h)e^{-x\sqrt{h}}\right]\cdot\left[C(h)e^{y\sqrt{-h}} + D(h)e^{-y\sqrt{-h}}\right]\, dh}$$

wobei $A(h),\ B(h)\ C(h),\ D(h)$ beliebige $\mathcal{R}$-integrierbare Funktionen seien können.

Aufgabe 04

Wir machen den gleichen Ansatz $u(x,y) = f(x) \cdot g(y)$, gehen damit analog wie vorhin in die Differentialgleichung ein:

$$0 = \Box u = g \cdot \frac{\partial^2 f}{\partial x^2} - f \cdot \frac{\partial^2 g}{\partial y^2} = g \cdot \frac{d^2 f}{dx^2} - f \cdot \frac{d^2 g}{dy^2}$$

$$\rightarrow \ \frac{1}{f} \cdot \frac{d^2 f}{dx^2} = \frac{1}{g} \cdot \frac{d^2 g}{dy^2} =: h : const$$

und erhalten

$$f_h(x) = A_h e^{x\sqrt{h}} + B_h e^{-x\sqrt{h}} \quad : h \neq 0$$

$$g_h(y) = C_h e^{y\sqrt{h}} + D_h e^{-y\sqrt{h}} \quad : h \neq 0$$

$$f_0(x) = A_0 + B_0 x, \ g_0(y) = C_0 + D_0 y$$

Die allgemeine Lösung ergibt sich demnach als

$$\boxed{u(x,y) = (A(0) + B(0)x) \cdot (C(0) + D(0)y) + \int\limits_{\mathbb{R}\backslash\{0\}} \left[A(h)e^{x\sqrt{h}} + B(h)e^{-x\sqrt{h}}\right] \cdot \left[C(h)e^{y\sqrt{h}} + D(h)e^{-y\sqrt{h}}\right] \ dh}$$

wobei $A(h), \ B(h) \ C(h), \ D(h)$ beliebige $\mathcal{R}$-integrierbare Funktionen seien können.

Aufgabe 05

Sei $u : \mathbb{R}^n \rightarrow \mathbb{R}$ eine harmonische Funktion, das heisst

$$\Delta u = \sum_k \frac{\partial^2 u}{\partial x_k^2} = 0$$

Dann folgt für die jeweils betrachtete Funktion γ

a) Für $h = \sum_j h_j \vec{e}_j$ ist $\gamma = u(x+h)$ harmonisch, denn

$$\Delta u\,(x+h) = \sum_k \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} u\,(x+h) = \sum_k \frac{\partial}{\partial x_k} \sum_j \frac{\partial u(x+h)}{\partial (x_j + h_j)} \underbrace{\frac{\partial (x_j + h_j)}{\partial x_k}}_{\delta_{jk}} = \sum_k \frac{\partial}{\partial x_k} \frac{\partial u(x+h)}{\partial (x_k + h_k)}$$

$$= \sum_k \sum_j \underbrace{\frac{\partial (x_j + h_j)}{\partial x_k}}_{\delta_{jk}} \cdot \frac{\partial}{\partial (x_j + h_j)} \frac{\partial u(x+h)}{\partial (x_k + h_k)} = \sum_k \frac{\partial^2 u(x+h)}{\partial (x_k + h_k)^2} = 0$$

b) Die Funktion $u(\lambda x)$ ist ebenfalls harmonisch:

$$\Delta u(\lambda x) = \sum_k \frac{\partial}{\partial x_k} \frac{\partial u(\lambda x)}{\partial x} = \sum_k \frac{\partial}{\partial x_k} \sum_j \frac{\partial u(\lambda x)}{\partial (\lambda x_j)} \cdot \underbrace{\frac{\partial (\lambda x_j)}{\partial x_k}}_{\lambda \delta_{jk}} = \lambda \sum_k \frac{\partial}{\partial x_k} \frac{\partial u(\lambda x)}{\partial (\lambda x_k)}$$

$$= \lambda \sum_k \sum_j \underbrace{\frac{\partial (\lambda x_j)}{\partial x_k}}_{\lambda \delta_{jk}} \frac{\partial}{\partial (\lambda x_k)} \frac{\partial u(\lambda x)}{\partial (\lambda x_k)} = \lambda^2 \sum_k \frac{\partial^2 u(\lambda x)}{\partial (\lambda x_k)^2} = 0$$

c) Für die inverse $\hat{C}$ gilt $\hat{C} = C^T$. Bezeichnen mit C^i_j die Komponente in der i-ten Zeile und j-te Spalte, und schreiben $C^j_i x^i := \sum_i C^j_i x_i$

$$\Delta u(Cx) = \sum_k \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k} u(Cx) = \sum_k \frac{\partial}{\partial x_k} \sum_j \frac{\partial u(Cx)}{\partial (Cx)_j} \cdot \frac{\partial (Cx)_j}{\partial x_k} = \sum_k \frac{\partial}{\partial x_k} \sum_j \frac{\partial u(Cx)}{\partial (Cx)_j} \cdot \underbrace{\frac{\partial (C^j_i x^i)}{\partial x_k}}_{C^j_i \delta^i_k}$$

$$= \sum_k \sum_j C^j_k \frac{\partial}{\partial x_k}\frac{\partial u(Cx)}{\partial (Cx)_j} = \sum_k \sum_j C^j_k \sum_l \frac{\partial}{\partial (Cx)_l}\frac{\partial u(Cx)}{\partial (Cx)_j} \cdot \frac{\partial (Cx)_l}{\partial x_k} = \sum_k \sum_j \sum_l C^j_k \left(u''(Cx)\right)^l_j \cdot \underbrace{\frac{\partial (C^l_m x^m)}{\partial x_k}}_{C^l_m \delta^m_k}$$

$$= \sum_k \sum_j \sum_l C^j_k C^l_k \left(u''(Cx)\right)^l_j = \sum_k \sum_l C^l_k \left(u'' \cdot C\right)^l_k = \sum_k \sum_l \hat{C}^k_l \left(u'' \cdot C\right)^l_k = \sum_l \left[(u'' \cdot C) \cdot \hat{C}\right]^l_l$$

$$= \sum_l \left(u''\right)^l_l = \Delta u = 0$$

Somit ist auch $u(Cx)$ harmonisch.

d) Siehe nächsten Fall.

e) Allgemein gilt für $n \in \mathbb{N}$

$$\Delta \underbrace{\left(\frac{\partial u}{\partial x_1} \cdot \frac{\partial u}{\partial x_2}\right)}_{\gamma} = \sum_k \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_1} \cdot \frac{\partial u}{\partial x_2}\right) = \sum_k \frac{\partial}{\partial x_k}\left(\frac{\partial^2 u}{\partial x_k \partial x_1}\frac{\partial u}{\partial x_2} + \frac{\partial^2 u}{\partial x_k \partial x_2}\frac{\partial u}{\partial x_1}\right)$$

$$= \sum_k \left\{\frac{\partial^3 u}{\partial x_k^2 \partial x_1}\frac{\partial u}{\partial x_2} + \frac{\partial^2 u}{\partial x_k \partial x_1}\frac{\partial^2 u}{\partial x_k \partial x_2} + \frac{\partial^3 u}{\partial x_k^2 \partial x_2}\frac{\partial u}{\partial x_1} + \frac{\partial^2 u}{\partial x_k \partial x_2}\frac{\partial^2 u}{\partial x_k \partial x_1}\right\}$$

$$= \sum_k \left\{\frac{\partial u}{\partial x_2}\frac{\partial}{\partial x_1}\frac{\partial^2 u}{\partial x_k^2} + 2\frac{\partial^2 u}{\partial x_k \partial x_1}\frac{\partial^2 u}{\partial x_k \partial x_2} + \frac{\partial u}{\partial x_1}\frac{\partial}{\partial x_2}\frac{\partial^2 u}{\partial x_k^2}\right\}$$

$$= \frac{\partial u}{\partial x_2}\frac{\partial}{\partial x_1}\underbrace{\sum_k \frac{\partial^2 u}{\partial x_k^2}}_{0} + \frac{\partial u}{\partial x_1}\frac{\partial}{\partial x_2}\underbrace{\sum_k \frac{\partial^2 u}{\partial x_k^2}}_{0} + 2\sum_k \frac{\partial^2 u}{\partial x_k \partial x_1}\frac{\partial^2 u}{\partial x_k \partial x_2} = 2\sum_k \frac{\partial^2 u}{\partial x_k \partial x_1}\frac{\partial^2 u}{\partial x_k \partial x_2}$$

Für den Fall $n = 2$ ist also γ harmonisch:

$$\Delta\gamma = \underbrace{2\left[\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2 \partial x_1}\frac{\partial^2 u}{\partial x_2^2}\right]}_{2\frac{\partial^2 u}{\partial x_1 \partial x_2} \cdot \Delta u = 0} = 0$$

Für $n > 2$ ist ferner γ allgemein nicht harmonisch!

f) Auch hier ist γ harmonisch. Siehe dazu nächsten Fall.

g) Allgemein für $n \in \mathbb{N}$ und $a_i \in \mathbb{R}$ gilt

$$\Delta \sum_i a_i x_i \frac{\partial u}{\partial x_i} = \sum_k \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\sum_i a_i x_i \frac{\partial u}{\partial x_i} = \sum_k \frac{\partial}{\partial x_k}\sum_i a_i \left\{ x_i \frac{\partial^2 u}{\partial x_k \partial x_i} + \underbrace{\frac{\partial x_i}{\partial x_k}}_{\delta_{ik}} \frac{\partial u}{\partial x_i} \right\} = \sum_k \frac{\partial}{\partial x_k}\left[\sum_i a_i x_i \frac{\partial^2 u}{\partial x_k \partial x_i} + a_k \frac{\partial u}{\partial x_k}\right]$$

$$= \sum_k \left\{ \sum_i a_i \underbrace{\frac{\partial x_i}{\partial x_k}}_{\delta_{ik}} \frac{\partial^2 u}{\partial x_k \partial x_i} + \sum_i a_i x_i \frac{\partial^3 u}{\partial x_k^2 \partial x_i} + a_k \frac{\partial^2 u}{\partial x_k^2} \right\} = \sum_k a_k \frac{\partial^2 u}{\partial x_k^2} + \sum_k \sum_i a_i x_i \frac{\partial^3 u}{\partial x_k^2 \partial x_i} + \sum_k a_k \frac{\partial^2 u}{\partial x_k^2}$$

$$= 2 \cdot \sum_k a_k \frac{\partial^2 u}{\partial x_k^2} + \sum_i a_i x_i \frac{\partial}{\partial x_i} \underbrace{\sum_k \frac{\partial^2 u}{\partial x_k^2}}_{\Delta u} = 2 \cdot \sum_k a_k \frac{\partial^2 u}{\partial x_k^2}$$

Für den Fall $n \in \mathbb{N},\ a_i = a_j = 1\ \forall\ i, j$ (vorige Aufgabe) ist γ demnach harmonisch:

$$\Delta\gamma = 2 \underbrace{\sum_{k=1}^{2} \frac{\partial^2 u}{\partial x_k^2}}_{\Delta u} = 0$$

Für den Fall $n = 2,\ a_1 = -a_2 = 1$ ist γ allgemein <u>nicht</u> harmonisch:

$$\Delta\gamma = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}$$

h) Die Funktion γ ist harmonisch:

$$\Delta\gamma = \sum_k \frac{\partial}{\partial x_k}\left[\underbrace{\frac{\partial x_2}{\partial x_k}}_{\delta_{k2}} \frac{\partial u}{\partial x_1} + x_2 \frac{\partial^2 u}{\partial x_k \partial x_1} - \underbrace{\frac{\partial x_1}{\partial x_k}}_{\delta_{k1}} \frac{\partial u}{\partial x_2} - x_1 \frac{\partial^2 u}{\partial x_k \partial x_2} \right]$$

$$= \frac{\partial}{\partial x_2}\frac{\partial u}{\partial x_1} - \frac{\partial}{\partial x_1}\frac{\partial u}{\partial x_2} + \sum_k \left\{ \underbrace{\frac{\partial x_2}{\partial x_k}}_{\delta_{k2}} \frac{\partial^2 u}{\partial x_k \partial x_1} + x_2 \frac{\partial^3 u}{\partial x_k^2 \partial x_1} - \underbrace{\frac{\partial x_1}{\partial x_k}}_{\delta_{k1}} \frac{\partial^2 u}{\partial x_k \partial x_2} - x_1 \frac{\partial^3 u}{\partial x_k^2 \partial x_2} \right\}$$

$$= \frac{\partial^2 u}{\partial x_2 \partial x_1} - \frac{\partial^2 u}{\partial x_1 \partial x_2} + \sum_k \left\{ x_2 \frac{\partial^3 u}{\partial x_k^2 \partial x_1} - x_1 \frac{\partial^3 u}{\partial x_k^2 \partial x_2} \right\} = x_2 \frac{\partial}{\partial x_1} \underbrace{\sum_k \frac{\partial^2 u}{\partial x_k^2}}_{\Delta u} - x_1 \frac{\partial}{\partial x_2} \underbrace{\sum_k \frac{\partial^2 u}{\partial x_k^2}}_{\Delta u} = 0$$

i) Siehe nächsten Fall.

j) Für $n \in \mathbb{N}$ und $a_i \in \mathbb{R}$ gilt allgemein

$$\Delta \sum_i a_i \left(\frac{\partial u}{\partial x_i}\right)^2 = \sum_k \sum_i a_i \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_i}\right)^2 = \sum_k \sum_i a_i \frac{\partial}{\partial x_k}\left[2 \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_k \partial x_i}\right]$$

$$= \sum_{k,i} 2a_i \left\{ \frac{\partial^2 u}{\partial x_k \partial x_i}\frac{\partial^2 u}{\partial x_k \partial x_i} + \frac{\partial u}{\partial x_i}\frac{\partial^3 u}{\partial x_k^2 \partial x_i} \right\} = 2 \sum_{k,i} a_i \left(\frac{\partial^2 u}{\partial x_k \partial x_i}\right)^2 + 2 \sum_i a_i \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i} \underbrace{\sum_k \frac{\partial^2 u}{\partial x_k^2}}_{\Delta u} = 2 \sum_{k,i} a_i \left(\frac{\partial^2 u}{\partial x_k \partial x_i}\right)^2$$

Speziell für $n = 2,\ a_1 = -a_2 = 1$ ist γ harmonisch:

$$\Delta\gamma = 2\sum_k \left\{ \left(\frac{\partial^2 u}{\partial x_k \partial x_1}\right)^2 - \left(\frac{\partial^2 u}{\partial x_k \partial x_2}\right)^2 \right\} = 2\left\{ \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 - \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 \right\} = 2\left\{ \left(-\frac{\partial^2 u}{\partial x_2^2}\right)^2 - \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 \right\} = 0$$

Für $n = 2,\ a_1 = a_2 = 1$ ist jedoch γ allgemein <u>nicht</u> harmonisch:

$$\Delta\gamma = 2\sum_k \left\{ \left(\frac{\partial^2 u}{\partial x_k \partial x_1}\right)^2 + \left(\frac{\partial^2 u}{\partial x_k \partial x_2}\right)^2 \right\} = 2\left\{ \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2 \right\} \geq 0$$