Übungen zur Analysis II
FSU Jena - SS 07
Serie 13 - Lösungen

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Aufgabe 01

a)

$$s = \int_0^1 \left|\dot{\vec{r}}\right| dt = \int_0^1 \sqrt{9 + 36t^2 + 36t^4}\ dt = \int_0^1 (3 + 6t^2) dt = 5$$

b)

$$s = \lim_{b \to \infty} \int_0^b \left|\dot{\vec{r}}\right|\ dt = \int_0^\infty e^{-t} \cdot \sqrt{2\cos^2 t + 2\sin^2 t + 1}\ dt = \sqrt{3} \cdot \int_0^\infty e^{-t} dt = \sqrt{3}$$

c)

$$\vec{r} = \rho \cdot \cos\varphi \cdot \vec{e}_x + \rho \cdot \sin\varphi \cdot \vec{e}_y \ \rightarrow\ \frac{d}{d\varphi}\vec{r} = (\rho' \cos\varphi - \rho \sin\varphi) \cdot \vec{e}_x + (\rho' \sin\varphi + \rho\cos\varphi) \cdot \vec{e}_y$$

$$s = \int_0^{2\pi} \left|\dot{\vec{r}}\right|\ d\varphi = \int_0^{2\pi} \sqrt{\rho'^2 + \rho^2}\ d\varphi = 2c \cdot \int_0^\pi \sqrt{2 + 2\cos\varphi}\ d\varphi = 8c \quad (Sub:\ \ u := \cos\varphi)$$

d)

$$(x-y)^2 = a(x+y),\ x^2 - y^2 = \frac{9z^2}{8} \ \rightarrow\ (x-y)(x+y) = \frac{(x-y)^3}{a} = \frac{9z^2}{8} \ \rightarrow\ x = \left(\frac{9az^2}{8}\right)^{1/3} + y$$

$$\rightarrow\ y^2 - x^2 = y^2 - \left(\frac{9az^2}{8}\right)^{2/3} - y^2 - 2y\left(\frac{9az^2}{8}\right)^{1/3} = \frac{9z^2}{8} \ \rightarrow\ y = -\left(\frac{9z^2}{8}\right)^{2/3} \cdot \frac{1}{2\sqrt[3]{a}} - \frac{1}{2} \cdot \left(\frac{9az^2}{8}\right)^{1/3}$$

$$\rightarrow\ x = \frac{1}{2} \cdot \left(\frac{9az^2}{8}\right)^{1/3} - \left(\frac{9z^2}{8}\right)^{2/3} \cdot \frac{1}{2\sqrt[3]{a}}$$

$$\vec{r} := x\vec{e}_x + y\vec{e}_y + t\vec{e}_z\ ,\ t := z,\ \rightarrow\ \dot{\vec{r}} = \left(\frac{1}{2}\sqrt[3]{\frac{a}{3}} \cdot z^{-\frac{1}{3}} - \sqrt[3]{\frac{3}{a}} \cdot z^{\frac{1}{3}}\right) \cdot \vec{e}_x - \left(\frac{1}{2}\sqrt[3]{\frac{a}{3}} \cdot z^{-\frac{1}{3}} + \sqrt[3]{\frac{3}{a}} \cdot z^{\frac{1}{3}}\right) \cdot \vec{e}_y + \vec{e}_z$$

$$s = \int_0^{z_0} \left|\dot{\vec{r}}\right| \cdot dt = \int_0^{z_0} \sqrt{\frac{1}{2} \cdot \left(\frac{a}{3}\right)^{2/3} \cdot t^{-\frac{2}{3}} + 2\left(\frac{3}{a}\right)^{2/3} \cdot t^{\frac{2}{3}} + 1} \cdot dt$$

e)

$$\dot{\vec{r}} = r(1-\cos t)\cdot\vec{e}_x + r\sin t\cdot\vec{e}_y$$

$$s = \int_0^{2\pi} \left|\dot{\vec{r}}\right| \, dt = r\cdot\int_0^{2\pi}\sqrt{2-2\cos t}\, dt \overset{*}{=} 2r\sqrt{2}\cdot\int_0^{\pi}\sqrt{1-\cos t}\, dt \overset{**}{=} 2r\sqrt{2}\left[-2\sqrt{1+\cos t}\right]_0^{\pi} = 8r$$

$$* Sub: u = t-2\pi \quad **Sub:\ u = \cos\varphi$$

Aufgabe 03

$$x_0 = \frac{1}{l}\int_0^{\pi} r(t-\sin t)\cdot r\sqrt{2-2\cos t}\, dt = \frac{r}{2\sqrt{2}}\int_0^{\pi}(t-\sin t)\sqrt{1-\cos t}\, dt$$

$$= \frac{r}{2\sqrt{2}}\cdot\left[-2t\sqrt{1+\cos t}+4\sqrt{1-\cos t}-\frac{2}{3}(1-\cos t)^{3/2}\right]_0^{\pi} = \frac{4r}{3}$$

$$y_0 = \frac{1}{4r}\int_0^{\pi} r(1-\cos t)\cdot r\sqrt{2-2\cos t}\, dt = \frac{r}{2\sqrt{2}}\int(1-\cos t)^{3/2}\, dt$$

$$= \frac{r}{2\sqrt{2}}\cdot\left[-4\sqrt{1+\cos t}+\frac{2}{3}\cdot(1+\cos t)^{3/2}\right]_0^{\pi} = \frac{4r}{3} \quad (Sub: u := \cos t)$$

Aufgabe 04

a) Nennen $\vec{r} := x$ und re-definieren $x := x_1, y := y_1$. Der Weg γ ist beschrieben durch

$$\vec{r} = x\vec{e}_x + (\pi - x)\vec{e}_y,\ x\in(0,\pi) \ \rightarrow\ d\vec{r} = dx\vec{e}_x - dx\vec{e}_y$$

weshalb sich das Kurvenintegral ergibt als

$$\int_\gamma \vec{f}(\vec{r})\cdot d\vec{r} = \int_\gamma(\sin x - \sin y(x))\, dx = \int_0^{\pi}\underbrace{(\sin x - \sin(\pi - x))}_{0}\, dx = 0$$

b) Die 3 Teilwege $\gamma_1, \gamma_2, \gamma_3$ sind beschrieben durch

$$\gamma_1 : \vec{r} = \cos t\cdot\vec{e}_x + \sin t\cdot\vec{e}_y \qquad \gamma_2 : \vec{r} = \cos t\cdot\vec{e}_y + \sin t\cdot\vec{e}_z \qquad \gamma_3 : \vec{r} = \cos t\cdot\vec{e}_z + \sin t\cdot\vec{e}_x$$

wobei t jeweils durch $[0, \pi/2]$ läuft. Das Kurvenintegral S über die geschlossene Kurve γ ergibt sich also

$$S = \oint_\gamma \vec{f} \cdot d\vec{r} = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \right) \left(\vec{f} \cdot d\vec{r} \right)$$

$$= \int_{\gamma_1} \begin{pmatrix} \sin^2 t - \cos^2 t \\ -\cos^2 t \\ \cos^2 t - \sin^2 t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt + \int_{\gamma_2} \begin{pmatrix} \cos^2 t \\ \sin^2 t \\ -\cos^2 t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} dt + \int_{\gamma_3} \begin{pmatrix} -\sin^2 t \\ \cos^2 t - \sin^2 t \\ \sin^2 t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ 0 \\ -\sin t \end{pmatrix} dt$$

$$= \int_0^{\pi/2} \left(\cos^2 t \sin t - \cos t \sin^2 t - 3 \sin^3 t - 2 \cos^3 t \right) dt$$

$$= \int_0^{\pi/2} \left(\cos^2 t \sin t - \cos t \sin^2 t - 3(1 - \cos^2 t) \sin t - 2(1 - \sin^2 t) \cos t \right) dt = \int_0^{\pi/2} \left(\sin^2 t \cos t + 4 \cos^2 \sin t - 2 \cos t - 3 \sin t \right) dt$$

$$= \left[\frac{\sin^3 t - 4 \cos^3 t}{3} - 2 \sin t + 3 \cos t \right]_0^{\pi/2} = -\frac{10}{3}$$