Übungen zur Analysis II
FSU Jena - SS 07
Serie 09 - Lösungen

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Aufgabe 01

a) Sei I das Definitionsgebiet von f.

$$\vec{\nabla} f = (\cos x - \cos(x+y+z), \cos y - \cos(x+y+z), \cos z - \cos(x+y+z)) \overset{!}{=} \vec{0} \rightsquigarrow \cos x = \cos y = \cos z = \cos(x+y+z)$$

$$\Rightarrow (x,y,z) \in \left\{(0,0,0),\ \frac{\pi}{2}(1,1,1),\ (\pi,\pi,\pi)\right\}$$

$$H(x,y,z) := f''(x,y,z) = \begin{pmatrix} -\sin x + \sin(x+y+z) & \sin(x+y+z) & \sin(x+y+z) \\ \sin(x+y+z) & -\sin y + \sin(x+y+z) & \sin(x+y+z) \\ \sin(x+y+z) & \sin(x+y+z) & -\sin z + \sin(x+y+z) \end{pmatrix}$$

$$H(\pi,\pi,\pi) = H(0,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow nicht\ definit \rightarrow keine\ Aussage$$

$$Doch\ \exists\, \varepsilon > 0 : \forall\, \vec{r} \in B_\varepsilon(0) \cap I : (f'(\vec{r}))_i \geq 0 \Rightarrow Lokales\ Minimum\ da\ f'(0)\cdot\vec{r} \geq 0$$

$$Analog\ \exists\, \varepsilon > 0 : \forall\, \vec{r} \in B_\varepsilon(\vec{\pi}) \cap I : (f'(\vec{r}))_i \leq 0 \Rightarrow Lokales\ Minimum\ da\ f'(\vec{\pi})(\vec{r}-\vec{\pi}) \geq 0$$

$$H\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right) = \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix} \rightsquigarrow P_\lambda(H) = (\lambda+1)^2(x+4)$$

$$\rightarrow Eigenwerte:\ \lambda_{1,2} = -1, -4 \rightarrow H\ negativ\ definit \rightarrow Isoliertes\ lokales\ Maximum$$

b) **Definieren:** $x_0 := a,\ x_{n+1} := b,\ \vec{r} := (r_1, ..., r_n) \in \mathbb{R}^n,\ r_0 := r_{n+1} := 0$ und $\alpha \in \mathbb{R}^n$ die gesuchte extrem-Stelle.

$$w := \ln u = \sum_{i=1}^{n} \ln x_i - \sum_{i=0}^{n} \ln(x_i + x_{i+1})$$

$$\partial_{x_i} w = \frac{1}{x_i} - \frac{1}{x_{i-1} + x_i} - \frac{1}{x_i + x_{i+1}} \overset{!}{=} 0,\ i = 1, ..., n \ \rightsquigarrow\ x_i^2 = x_{i-1} x_{i+1}$$

$$Induktionsanfang:\ x_2 = \frac{x_1^2}{a}\ (Klar)$$

$$Induktionsannahme:\ x_k = \frac{x_1^k}{a^{k-1}}\ \ \forall\ k = 1, .., n, n+1$$

$$Induktionsschritt:\ x_{k+1} = \frac{x_k^2}{x_{k-1}} = \frac{\left(\frac{x_1^k}{a^{k-1}}\right)^2}{\frac{x_1^{k-1}}{a^{k-2}}} = \frac{x_1^{k+1}}{a^k}$$

$$\Rightarrow\ x_{n+1} = b = \frac{x_1^{n+1}}{a^n}\ \Rightarrow\ x_1 = \sqrt[n+1]{ba^n},\ x_k = \frac{(ba^n)^{\frac{k}{n+1}}}{a^{k-1}} \geq 0,\ k = 1, ..., n$$

$$\partial_{i^2} w = -\frac{1}{x_i^2} + \frac{1}{(x_{i-1} + x_i)^2} + \frac{1}{(x_i + x_{i+1})^2},\ \partial_{i-1,i} w = \frac{1}{(x_{i-1} + x_i)^2},\ \partial_{i+1,i} w = \frac{1}{(x_i + x_{i+1})^2}$$

$$Definieren:\ \partial_{0,1} w = \partial_{n+1,n} w = 0$$

$$\Omega(\vec{r}) :=< w''(\alpha)\vec{r}, \vec{r} >= \sum_{i=1}^{n} \left\{ \partial_{i-1,i} w \cdot r_{i-1} r_i + \partial_{i^2} w \cdot r_i^2 + \partial_{i+1,i} w \cdot r_{i+1} r_i \right\}$$

$$= \sum_{i=1}^{n} \left\{ 2\partial_{i,i+1} w \cdot r_i r_{i+1} + \partial_{i^2} w \cdot r_i^2 \right\} = \sum_{i=1}^{n} \left\{ \frac{2r_{i+1} r_i}{(x_i + x_{i+1})^2} + r_i^2 \cdot \left(\frac{1}{(x_{i-1} + x_i)^2} + \frac{1}{(x_i + x_{i+1})^2} - \frac{1}{x_i^2} \right) \right\}$$

$$= \sum_{i=1}^{n} \left\{ \frac{2 \cdot a^{2i} \cdot r_i r_{i+1}}{x_1^{2i}(a + x_1)^2} + r_i^2 \cdot \left(\frac{a^{2i-2}}{x_1^{2i-2}(a + x_1)^2} + \frac{a^{2i}}{x_1^{2i}(a + x_1)^2} - \frac{a^{2i-2}}{x_1^{2i}} \right) \right\}$$

$$= \sum_{i=1}^{n} \frac{a^{2i-2}}{x_1^{2i}(a + x_1)^2} \cdot \left\{ 2a^2 r_i r_{i+1} + r_i^2 \cdot \left(x_1^2 + a^2 - (a + x_1)^2 \right) \right\} = \sum_{i=1}^{n} \frac{a^{2i-2}}{x_1^{2i}(a + x_1)^2} \cdot \left\{ 2a^2 r_i r_{i+1} - 2a x_1 \cdot r_i^2 \right\}$$

$$\Omega(\vec{e}_i) = -\frac{2x_1^{1-2i} a^{2i-1}}{(a + x_1)^2} < 0\ \ \forall\ i = 1, ..., n\ \Rightarrow\ \forall\ \vec{r} \in \mathbb{R}^n \setminus \{0\}\ \exists\ \lambda_1, ..., \lambda_n \in \mathbb{R} : \vec{r} = \sum_{i=1}^{n} \lambda_i \cdot \vec{e}_i$$

$$\Rightarrow\ \Omega(\vec{r}) = \Omega\left(\sum_{i=1}^{n} \lambda_i \cdot \vec{e}_i \right) = \sum_{i,j=1}^{n} \{ \lambda_i \lambda_j \cdot < w''(\alpha)\vec{e}_i, \vec{e}_j > \} = \sum_{i,j=1}^{n} \{ \lambda_i \lambda_j \cdot \partial_{ij} w \} < 0$$

$$\rightarrow\ f''(a)\ negativ\ definit \Rightarrow\ Isoliertes,\ lokales\ Maximum$$

Bemerkung: $a, b \neq 0$. Oberes Extremum ist übrigens auch Extremum von $u()$.

c) **Definieren:** $x_0 := 1,\ x_{n+1} := 2,\ \vec{r} := (r_1, ..., r_n) \in \mathbb{R}^n,\ r_0 := r_{n+1} := 0$ und $a \in \mathbb{R}^n$ die gesuchte extrem-Stelle.

$$w(x_1, ..., x_n) = \sum_{i=1}^{n+1} \frac{x_i}{x_{i-1}},\ \partial_i w = \frac{1}{x_{i-1}} - \frac{x_{i+1}}{x_i^2} \overset{!}{=} 0,\ k = 1, ..., n \ \Rightarrow\ x_i^2 = x_{i-1}x_{i+1} \ \overset{b}{\Rightarrow}\ x_k = x_1^k$$

$$\Rightarrow\ x_{n+1} = x_1^{n+1} = 2 \ \Rightarrow\ x_1 = \sqrt[n+1]{2} \ \wedge\ x_k = 2^{\frac{k}{n+1}}$$

$$\partial_{i^2} w = \frac{2x_{i+1}}{x_i^3} = 2 \cdot 2^{\frac{1-2i}{n+1}},\ \partial_{i-1,i} w = -\frac{1}{x_{i-1}^2} = -2^{\frac{2-2i}{n+1}},\ \partial_{i+1,i} w = -\frac{1}{x_i^2} = -2^{-\frac{2i}{n+1}}$$

$$Definieren:\ \partial_{0,1} w = \partial_{n+1,n} w = 0$$

$$\Omega(\vec{r}) :=< w''(a)\vec{r}, \vec{r} >= \sum_{i=1}^{n} \left\{\partial_{i-1,i} w \cdot r_{i-1} r_i + \partial_{i^2} w \cdot r_i^2 + \partial_{i+1,i} w \cdot r_{i+1} r_i\right\} = \sum_{i=1}^{n} \left\{2\partial_{i+1,i} w \cdot r_i r_{r+1} + \partial_{i^2} w \cdot r_i^2\right\}$$

$$= 2 \cdot \sum_{i=1}^{n} \left\{-2^{\frac{-2i}{n+1}} \cdot r_{i+1} r_i + 2^{\frac{1-2i}{n+1}} \cdot r_i^2\right\} = 2 \cdot \sum_{i=1}^{n} 2^{\frac{-2i}{n+1}} \cdot \left\{2^{\frac{1}{n+1}} \cdot r_i^2 - r_{i+1} r_i\right\},\ \Omega:\ bilinear$$

$$\Omega(\vec{e}_i) = 2 \cdot 2^{\frac{1-2i}{n+1}} > 0 \ \ \forall\ i = 1, ..., n \ \Rightarrow\ \forall\ \vec{r} \in \mathbb{R}^n \setminus \{0\}\ \exists\ \lambda_1, ..., \lambda_n \in \mathbb{R} : \vec{r} = \sum_{i=1}^{n} \lambda_i \cdot \vec{e}_i$$

$$\Rightarrow\ \Omega(\vec{r}) = \Omega\left(\sum_{i=1}^{n} \lambda_i \cdot \vec{e}_i\right) = \sum_{i,j=1}^{n} \{\lambda_i \lambda_j \cdot < w''(a)\vec{e}_i, \vec{e}_j >\} = \sum_{i,j=1}^{n} \{\lambda_i \lambda_j \cdot \partial_{ij} w\} > 0$$

$$\rightarrow\ f''(a)\ positiv\ definit\ \rightarrow\ lokales\ Minimum$$

Bemerkung: Die Funktionswerte an den Extrema ergeben sich durch einfaches einsetzen.