Übungen zur Analysis II
FSU Jena - SS 07
Serie 06 - Lösungen

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Aufgabe 01

$u = x^2 e^x e^y + y^2 e^x e^y$

$$\frac{\partial u}{\partial y} = x^2 e^x e^y + y^2 e^x e^y + 2ye^x e^y = u + f,\ f := 2ye^x e^y,\ \frac{\partial f}{\partial y} = f + 2ye^x e^y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y} + f + 2e^x e^y,\ \frac{\partial^3 u}{\partial y^3} = \frac{\partial^2 u}{\partial y^2} + f + 4e^x e^y \ \Rightarrow\ \frac{\partial^n u}{\partial y^n} = \begin{cases} \dfrac{\partial^{n-1} u}{\partial u^{n-1}} + f + (n-1)\cdot 2e^x e^y & : n > 1 \\ u + f & : n = 1 \end{cases}$$

$$\Rightarrow\ \frac{\partial^n u}{\partial f^n} = u + n \cdot f + 2e^x e^y \cdot \sum_{i=2}^{n} (i-1) = u + 2n \cdot ye^x e^y + n(n-1) \cdot e^x e^y$$

$$Analog:\ \frac{\partial^m u}{\partial x^m} = x^2 e^x e^y + y^2 e^x e^y + 2m \cdot xe^x e^y + m(m-1) \cdot e^x e^y \ \Rightarrow\ \frac{\partial^{m+n} u}{\partial x^m \partial y^n} = \frac{\partial^m u}{\partial x^m} + \frac{\partial^m}{\partial x^m}\left(2n \cdot ye^x e^y + n(n-1) \cdot e^x e^y\right)$$

$$= \frac{\partial^m u}{\partial x^m} + [2n \cdot ye^x e^y + n(n-1) \cdot e^x e^y] = \left\{x^2 + y^2 + 2(mx + ny) + n(n-1) + m(m-1)\right\} \cdot e^x e^y$$

Aufgabe 02

$$x^2 = \frac{\xi+\eta}{2},\ y^2 = \frac{\xi-\eta}{2},\quad \frac{\partial u}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x} = \frac{\partial f}{\partial \xi}2x + \frac{\partial f}{\partial \eta}2x + \frac{\partial f}{\partial \zeta}2y$$

$$\Rightarrow\ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial \xi}2x + 2\frac{\partial f}{\partial \xi} + \frac{\partial^2 f}{\partial x \partial \eta}2x + 2\frac{\partial f}{\partial \eta} + \frac{\partial^2 f}{\partial x \partial \zeta}2y$$

$$= 2x\left(\frac{\partial^2 f}{\partial \xi^2}2x + \frac{\partial^2 f}{\partial \eta \partial \xi}2x + \frac{\partial^2 f}{\partial \zeta \partial \xi}2y\right) + 2x\left(\frac{\partial^2 f}{\partial \xi \partial \eta}2x + \frac{\partial^2 f}{\partial \eta^2}2x + \frac{\partial^2 f}{\partial \zeta \partial \eta}2y\right) + 2y\left(\frac{\partial^2 f}{\partial \xi \partial \zeta}2x + \frac{\partial^2 f}{\partial \eta \partial \zeta}2x + \frac{\partial^2 f}{\partial \zeta^2}2y\right) + 2\frac{\partial f}{\partial \xi} + 2\frac{\partial f}{\partial \eta}$$

$$= 2(\xi+\eta)\cdot\left[\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \xi \partial \eta} + \frac{\partial^2 f}{\partial \eta \partial \xi}\right] + 2\zeta\cdot\left[\frac{\partial^2 f}{\partial \zeta \partial \eta} + \frac{\partial^2 f}{\partial \eta \partial \zeta} + + \frac{\partial^2 f}{\partial \xi \partial \zeta} + \frac{\partial^2 f}{\partial \zeta \partial \xi}\right] + 2(\xi-\eta)\cdot\frac{\partial^2 f}{\partial \zeta^2} + 2\frac{\partial f}{\partial \xi} + 2\frac{\partial f}{\partial \eta}$$

$$Analog:\ \frac{\partial^2 u}{\partial y^2} = 2(\xi-\eta)\cdot\left[\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^2 f}{\partial \eta \partial \xi} - \frac{\partial^2 f}{\partial \xi \partial \eta}\right] + 2\zeta\cdot\left[\frac{\partial^2 f}{\partial \zeta \partial \xi} + \frac{\partial^2 f}{\partial \xi \partial \zeta} - \frac{\partial^2 f}{\partial \zeta \partial \eta} - \frac{\partial^2 f}{\partial \eta \partial \zeta}\right] + 2(\xi+\eta)\cdot\frac{\partial^2 f}{\partial \zeta^2} + 2\frac{\partial f}{\partial \xi} - 2\frac{\partial f}{\partial \eta}$$

Aufgabe 04

$$\partial_x f := \frac{\partial f}{\partial x} = \begin{cases} \dfrac{y^3(4x^2-y^2)+x^4 y}{(x^2+y^2)^2} & : (x,y) \neq (0,0)\ (direktes\ ableiten) \\ 0 & : (x,y) = (0,0)\ (Serie\ 04) \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h\to 0}\frac{\partial_x f(0,h) - \partial_x f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h^3(0-h^2)+0\cdot h}{(0^2+h^2)^2} - 0}{h} = -1$$

$$\partial_y f(x,y) := \frac{\partial f}{\partial y}(x,y) = \frac{\partial(-f)}{\partial y}(y,x) = -\partial_x f(y,x) = \begin{cases} \dfrac{x^3(x^2-4y^2)-y^4 x}{(x^2+y^2)^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0)\ (Serie\ 04) \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0}\frac{\partial_y f(h,0) - \partial_y f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h^3(h^2-0)-0\cdot h}{(h^2+0)^2} - 0}{h} = 1$$