Übungen zur Analysis II
FSU Jena - SS 07
Serie 05 - Lösungen

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Aufgabe 01

$$Id = (x)' = \left(f\left(f^{-1}(x)\right)\right)' = f'\left(f^{-1}(x)\right) \cdot \left(f^{-1}\right)'(x) \quad \square$$

Aufgabe 02

- $$\frac{\partial U}{\partial r} = \sin\theta\cos\varphi \cdot \frac{\partial u}{\partial x} + \sin\theta\sin\varphi \cdot \frac{\partial u}{\partial y} + \cos\theta \cdot \frac{\partial u}{\partial z}$$

- $$\frac{\partial^2 U}{\partial r^2} = \sin\theta\cos\varphi \cdot \left[\sin\theta\cos\varphi \cdot \frac{\partial^2 u}{\partial x^2} + \sin\theta\sin\varphi \cdot \frac{\partial^2 u}{\partial y\partial x} + \cos\theta \cdot \frac{\partial^2 u}{\partial z\partial x}\right]$$
  $$+ \sin\theta\sin\varphi \cdot \left[\sin\theta\cos\varphi \cdot \frac{\partial^2 u}{\partial x\partial y} + \sin\theta\sin\varphi \cdot \frac{\partial^2 u}{\partial y^2} + \cos\theta \cdot \frac{\partial^2 u}{\partial z\partial y}\right]$$
  $$+ \cos\theta \cdot \left[\cos\varphi\sin\theta \cdot \frac{\partial^2 u}{\partial x\partial z} + \sin\varphi\sin\theta \cdot \frac{\partial^2 u}{\partial y\partial z} + \cos\theta \cdot \frac{\partial^2 u}{\partial z^2}\right]$$

- $$\frac{\partial U}{\partial \varphi} = -r\sin\theta\sin\varphi \cdot \frac{\partial u}{\partial x} + r\sin\theta\cos\varphi \cdot \frac{\partial u}{\partial y}$$

- $$\frac{\partial^2 U}{\partial \varphi^2} = -r\sin\theta\cos\varphi \cdot \frac{\partial u}{\partial x} - r\sin\theta\sin\varphi \cdot \left[-r\sin\theta\sin\varphi \cdot \frac{\partial^2 u}{\partial x^2} + r\sin\theta\cos\varphi \cdot \frac{\partial^2 u}{\partial y\partial x}\right]$$
  $$- r\sin\theta\sin\varphi \cdot \frac{\partial u}{\partial y} + r\sin\theta\cos\varphi \cdot \left[-r\sin\theta\sin\varphi \cdot \frac{\partial^2 u}{\partial x\partial y} + r\sin\theta\cos\varphi \cdot \frac{\partial^2 u}{\partial y^2}\right]$$

- $$\frac{\partial U}{\partial \theta} = r\cos\theta\cos\varphi \cdot \frac{\partial u}{\partial x} + r\cos\theta\sin\varphi \cdot \frac{\partial u}{\partial y} - r\sin\theta\frac{\partial u}{\partial z}$$

- $$\frac{\partial^2 U}{\partial \theta^2} = -r\sin\theta\cos\varphi\cdot\frac{\partial u}{\partial x} + r\cos\theta\cos\varphi\cdot\left[r\cos\theta\cos\varphi\cdot\frac{\partial^2 u}{\partial x^2} + r\cos\theta\sin\varphi\cdot\frac{\partial^2 u}{\partial y\partial x} - r\sin\theta\cdot\frac{\partial^2 u}{\partial z\partial x}\right]$$

$$- r\sin\theta\sin\varphi\cdot\frac{\partial u}{\partial y} + r\cos\theta\sin\varphi\cdot\left[r\cos\theta\cos\varphi\cdot\frac{\partial^2 u}{\partial x\partial y} + r\cos\theta\sin\varphi\cdot\frac{\partial^2 u}{\partial y^2} - r\sin\theta\cdot\frac{\partial^2 u}{\partial z\partial y}\right]$$

$$- r\cos\theta\cdot\frac{\partial u}{\partial z} - r\sin\theta\cdot\left[r\cos\theta\cos\varphi\cdot\frac{\partial^2 u}{\partial x\partial z} + r\cos\theta\sin\varphi\cdot\frac{\partial^2 u}{\partial y\partial z} - r\sin\theta\cdot\frac{\partial^2 u}{\partial z^2}\right]$$

Obere Terme eingesetzt ergeben:

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial U}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial \varphi^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \Delta u \quad \square$$

Aufgabe 03

Es gelten die gleichen Bezeichnungen wie in Aufgabe (2).

$$f(x,y,z) = f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) = U(r,\theta,\varphi) = U(r) = \frac{1}{r}$$

$$\Rightarrow \Delta f = \frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial U}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial \varphi^2} = \frac{2}{r^3} - \frac{2}{r^3} = 0 \quad \square$$

Aufgabe 05

a)

$$\frac{\partial f}{\partial h} = \lim_{t\to 0}\frac{f\left(3+\frac{t}{\sqrt{2}}, 4+\frac{t}{\sqrt{2}}\right) - f(3,4)}{t} = \lim_{t\to 0}\frac{\sqrt{25+\frac{14t}{\sqrt{2}}+t^2}-5}{t} = \lim_{t\to 0}\frac{\frac{14}{\sqrt{2}}+t}{\sqrt{25+\frac{14t}{\sqrt{2}}+t^2}+5} = \frac{7}{5\sqrt{2}}$$

b)

$$\frac{\partial f}{\partial h} = \lim_{t\to 0}\frac{f\left(2+\frac{t}{2}, 2+\frac{t\sqrt{3}}{2}\right) - f(2,2)}{t} = \lim_{t\to 0}\frac{\sqrt{16+4(1+\sqrt{3})t+\sqrt{3}t^2}-4}{2t}$$

$$= \lim_{t\to 0}\frac{4+4\sqrt{3}+\sqrt{3}t}{2\cdot\left(\sqrt{16+4(1+\sqrt{3})t+\sqrt{3}t^2}+4\right)} = \frac{1+\sqrt{3}}{4}$$

Aufgabe 06

Sei $h = (h_x, h_y)$ eine beliebige Richtung. Dann

$$\frac{\partial f}{\partial h}(0,0) = \lim_{t\to 0}\frac{f(th_x, th_y) - f(0,0)}{t} = \lim_{t\to 0}\frac{|t|\cdot\sqrt{|h_x h_y|}}{t} = \sqrt{|h_x h_y|}\cdot\lim_{t\to 0} sgn(t) \;\to\; \begin{cases} = 0 & : h_x = 0 \vee h_y = 0 \\ \neg\exists & : h_x \neq 0 \neq h_y \end{cases}$$

Die Richtungsableitungen existieren also nur in Richtung der beiden Koordinaten, also muss gelten $h_x = 0 \vee h_y = 0$.