Übungen zur Vorlesung Analysis II - FSU Jena
SS 07 - Serie 01
- Lösungen -

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Aufgabe 1

a)

$$\int \frac{x^3+1}{x^3-5x^2+6x}dx = \int dx + \int \frac{5x^2-6x+1}{x^3-5x^2+6x}dx = x + \int \frac{1}{6} \cdot \left[\frac{1}{x} + \frac{56}{x-3} - \frac{27}{x-2}\right] dx$$

$$= x + \frac{1}{6} \cdot [\ln|x| + 56\ln|x-3| - 27\ln|x-2|] + C$$

b)

$$\int \frac{1}{x^3+3x^2-4}dx = \int \left[\frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}\right] dx = \frac{1}{9} \cdot \left[\ln|x-1| - \ln|x+2| + \frac{3}{x+2}\right] + C$$

c)

$$\int \frac{x^2-5}{(x-1)(x^2-2x+5)}dx = \int \left[\frac{-1}{x-1} + \frac{2x}{x^2-2x+5}\right] dx = -\ln(x-1) + \int \frac{2x-2}{x^2-2x+5}dx + \int \frac{2}{x^2-2x+5}dx$$

$$= -\ln|x-1| + \ln|x^2-2x+5| + \frac{1}{2} \cdot \int \frac{1}{\left(\frac{x-1}{2}\right)^2+1}dx = -\ln|x-1| + \ln\left|x^2-2x+5\right| + \arctan\left(\frac{x-1}{2}\right) + C$$

Aufgabe 2

$Sub: \ u := x + 2\pi$

$$A := \int_{-2\pi}^{2\pi} \frac{dx}{2\sin x - \cos x + 5} = \int_{-2\pi}^{0} \frac{dx}{2\sin x - \cos x + 5} + \int_{0}^{2\pi} \frac{dx}{2\sin x - \cos x + 5}$$

$$= \int_{-2\pi}^{0} \frac{dx}{2\sin(x+2\pi) - \cos(x+2\pi) + 5} + \int_{0}^{2\pi} \frac{dx}{2\sin x - \cos x + 5}$$

$$= \int_{0}^{2\pi} \frac{dx}{2\sin u - \cos u + 5} + \int_{0}^{2\pi} \frac{dx}{2\sin x - \cos x + 5} = 2\int_{0}^{2\pi} \frac{dx}{2\sin x - \cos x + 5}$$

$Sub: \ v := x - \pi$

$$\Rightarrow \ A = 2\left[\int_{0}^{\pi} \frac{dx}{2\sin x - \cos x + 5} + \int_{\pi}^{2\pi} \frac{dx}{2\sin x - \cos x + 5}\right] = 2\left[\int_{0}^{\pi} \frac{dx}{2\sin x - \cos x + 5} + \int_{\pi}^{2\pi} \frac{dx}{-2\sin(x-\pi) + \cos(x-\pi) + 5}\right]$$

$$= 2\left[\int_{0}^{\pi} \frac{dx}{2\sin x - \cos x + 5} + \int_{0}^{\pi} \frac{dv}{-2\sin v + \cos v + 5}\right] = 2\int_{0}^{\pi} \left[\frac{1}{2\sin x - \cos x + 5} + \frac{1}{-2\sin x + \cos x + 5}\right] dx$$

$$Sub: \ t := \tan\left(\frac{x}{2}\right) \ \rightarrow \ \sin x = \frac{2t}{1+t^2} \ \wedge \ \cos x = \frac{1-t^2}{1+t^2}$$

$$\Rightarrow \ A = 2\int_{0}^{\infty} \left[\frac{1}{3t^2 + 2t + 2} + \frac{1}{2t^2 - 2t + 3}\right] dt = 2\int_{0}^{\infty} \left[\frac{3}{5} \cdot \frac{1}{\left(\frac{3t+1}{\sqrt{5}}\right)^2 + 1} + \frac{2}{5} \cdot \frac{1}{\left(\frac{2t-1}{\sqrt{5}}\right)^2 + 1}\right] dt$$

$$= \frac{2}{\sqrt{5}} \cdot \left[\arctan\left[\frac{3t+1}{\sqrt{5}}\right] + \arctan\left[\frac{2t-1}{\sqrt{5}}\right]\right]_{0}^{\infty} = \frac{2\pi}{\sqrt{5}}$$

Aufgabe 3

a)

$$Sub: t := x^2: \int_{0}^{\infty} (-1)^{[x^2]} dx = \int_{0}^{\infty} \frac{(-1)^{[t]}}{2\sqrt{t}} dt = \lim_{N\to\infty} \int_{0}^{N} \frac{(-1)^{[t]}}{2\sqrt{t}} dt = \lim_{N\to\infty} \sum_{n=0}^{N-1} \int_{n}^{n+1} \frac{(-1)^{[t]}}{2\sqrt{t}} dt$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N-1} \int_{n}^{n+1} \frac{(-1)^n}{2\sqrt{t}} dt = \lim_{N\to\infty} \sum_{n=0}^{N-1} \left[(-1)^n \int_{n}^{n+1} \frac{1}{2\sqrt{t}} dt\right] = \sum_{n=0}^{\infty} \cdot (-1)^n \left[\sqrt{n+1} - \sqrt{n}\right]$$

$$Da \ (a_n) := \left(\sqrt{n+1} - \sqrt{n}\right) \rightarrow \ 0 \ \wedge \ monoton \ fallend, \ ist \ \sum_{n=0}^{\infty} (-1)^n a_n \ \ konvergent \ nach \ Leipniz \quad \square$$

b)

$$Annahme: n, m \in \mathbb{N}$$

$$Fall\ 1: \ n \leq m+1$$

$$\lim_{N\to\infty} \int_0^N \frac{x^m}{1+x^n}dx \geq \lim \int_0^N \frac{x^m}{1+x^{m+1}}dx = \int_0^1 \frac{x^m}{1+x^{m+1}}dx + \lim \int_1^N \frac{x^m}{1+x^{m+1}}dx$$

$$\geq \int_0^1 \frac{x^m}{1+x^{m+1}}dx + \lim \int_1^N \frac{x^m}{2x^{m+1}}dx = \int_0^1 \frac{x^m}{1+x^{m+1}}dx + \lim_{N\to\infty} \frac{\ln(N)}{2} = \infty \ \Rightarrow \ Nicht\ konvergent$$

$$Fall\ 2: \ n > m+1$$

$$f(N) := \int_0^N \frac{x^m}{1+x^n}dx \ monoton\ wachsend\ da\ \frac{x^m}{1+x^n} \geq 0$$

$$Ausserdem: \ \lim_{N\to\infty} f(N) = \int_0^1 \frac{x^m}{1+x^n} + \int_1^\infty \frac{x^m}{1+x^n} \leq \int_0^1 \frac{x^m}{1+x^n} + \int_1^\infty \frac{x^m}{x^n}dx$$

$$= \int_0^1 \frac{x^m}{1+x^n} + \lim_{N\to\infty}\left[\frac{N^{m-n+1}}{m-n+1} - \frac{1}{m-n+1}\right] = \int_0^1 \frac{x^m}{1+x^n} - \frac{1}{m-n+1} \ \in \mathbb{R} \ \Rightarrow \ Konvergent$$

c)

$$\int_0^\infty \frac{\sin x}{x}dx = \lim_{N\to\infty}\sum_{n=0}^N \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}dx = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} (-1)^n \left|\frac{\sin x}{x}\right| dx = \sum_{n=0}^\infty (-1)^n \int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx$$

$$Sei\ a_n := \int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx > 0$$

$$Es\ gilt: (a_n) \to 0\ denn:$$

$$a_n \leq \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{n\pi}dx \overset{(1)}{=} \frac{|\cos(n\pi) - \cos((n+1)\pi)|}{n\pi} = \frac{2}{n\pi} \to 0 \ \ (Vergleichskriterium)$$

$$Ausserdem: \ (a_n)\ monoton\ fallend\ denn:$$

$$a_{n+1} = \int_{(n+1)\pi}^{(n+2)\pi} \frac{|\sin x|}{x}dx \leq \int_{(n+1)\pi}^{(n+2)\pi} \frac{|\sin x|}{(n+1)\pi}dx \leq \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}dx = a_n$$

$$\Rightarrow \sum_{n=0}^\infty (-1)^n a_n\ konvergent\ nach\ Leipniz \quad \square$$

*1 : Valid, da von $n\pi$ bis $(n+1)\pi$, $sgn(\sin x)$ immer konstant bleibt!

Aufgabe 4

a)

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+5} dx = \int_{-\infty}^{\infty} \frac{1}{(x+1)^2+4} dx = \frac{1}{4}\int_{-\infty}^{\infty} \frac{1}{(\frac{x+1}{2})^2+1} dx = \frac{1}{4}\left[\int_{-\infty}^{0} \frac{1}{(\frac{x+1}{2})^2+1} dx + \int_{0}^{\infty} \frac{1}{(\frac{x+1}{2})^2+1} dx\right]$$

$$= \frac{1}{2}\left[\lim_{N\to\infty}\left[\arctan\left(\frac{x+1}{2}\right)\right]_{-N}^{0} + \lim_{N\to\infty}\left[\arctan\left(\frac{x+1}{2}\right)\right]_{0}^{N}\right] = \frac{1}{2}\cdot\lim_{N\to\infty}\left[\arctan\left(\frac{N+1}{2}\right) - \arctan\left(\frac{-N+1}{2}\right)\right] = \frac{\pi}{2}$$

b)

$$\int_0^2 \frac{dx}{\sqrt{|1-x^2|}} = \int_0^1 \frac{dx}{\sqrt{|1-x^2|}} + \int_1^2 \frac{dx}{\sqrt{|1-x^2|}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \int_1^2 \frac{dx}{\sqrt{x^2-1}}$$

$$\left(Sub: t = x + \sqrt{x^2-1}\right)$$

$$= \lim_{h\nearrow 1}\left[\arcsin x\right]_0^h + \int_1^{2+\sqrt{3}} \frac{dt}{t} = \arcsin 1 - \arcsin 0 + \left[\ln t\right]_1^{2+\sqrt{3}} = \frac{\pi}{2} + \ln\left|2+\sqrt{3}\right| - \ln 1 = \frac{\pi}{2} + \ln\left|2+\sqrt{3}\right|$$