Analysis I - Serie 06
FSU Jena - WS 06/07
- Lösungen -

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Aufgabe 1

a)

$$a_n := \sin(n\lambda) \rightarrow \textit{ konvergent für } \lambda = k\pi,\ k \in \mathbb{Z}$$

b)

$$\lim a_n := \lim \frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n} = \frac{\sqrt[n]{n}}{\left(1+\frac{1}{n^2}\right)^n} = \lim \frac{1}{\sqrt[n]{\left(1+\frac{1}{n^2}\right)^{n^2}}} = \frac{1}{\lim \frac{\left(1+\frac{1}{(n+1)^2}\right)^{(n+1)^2}}{\left(1+\frac{1}{n^2}\right)^{n^2}}} = \frac{1}{\frac{e}{e}} = 1$$

$$\rightarrow \textit{ divergent}$$

c)

$$a_n := \frac{2+(-1)^n}{2^{n+1}}$$

$$\sqrt[n]{a_n} = \frac{1}{2} \cdot \sqrt[n]{1+\frac{(-1)^n}{2}} \leq \frac{1}{2} \cdot \sqrt[n]{1+\frac{1}{2}} \leq \frac{1}{2} \cdot \sqrt{\frac{3}{2}} < 1 \ \Rightarrow \textit{ konvergent}$$

d)

$$a_n := \frac{n^n \lambda^n}{n!}$$

$$\frac{a_{n+1}}{a_n} = \left(1+\frac{1}{n}\right)^n \cdot \lambda \rightarrow e\lambda$$

$$\Rightarrow \textit{ Konvergent für } \lambda < \frac{1}{e},\ \textit{divergent für } \lambda > \frac{1}{e}$$

$$\textit{Für } \lambda = \frac{1}{e} : \textit{Vorschläge willkommen!}$$

e)

$$a_n := \frac{\sqrt{n+1}-\sqrt{n-1}}{n} = \frac{2}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)} \leq \frac{2}{n\sqrt{n+1}} \leq \frac{2}{n\sqrt{n}} = \frac{2}{n^{3/2}} \ \rightarrow \textit{ konvergent}$$

f)

$$a_n := \left(\frac{1 \cdot 3 \cdot ... \cdot (2n-1)}{2 \cdot 4 \cdot .... \cdot 2n}\right)^{\lambda}$$

$$1 > 0 \;\Rightarrow\; (2n-1)^2 > (2n)(2n-2) \;\Rightarrow\; \frac{(2n-1)^2}{(2n-2)(2n)} > 1$$

$$3 < 4 \;\Rightarrow\; (2n-3)(2n-1) < (2n-2)^2 \;\Rightarrow\; \frac{(2n-3)(2n-1)}{(2n-2)^2} < 1$$

$$a_n = \left(\left(\frac{1 \cdot 3 \cdot ... \cdot (2n-1)}{2 \cdot 4 \cdot .... \cdot 2n}\right)^2\right)^{\lambda/2} = \left(\frac{1}{2} \cdot \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot ... \cdot \frac{(2n-1)^2}{(2n-2)(2n)} \cdot \frac{1}{2n}\right)^{\lambda/2} > \left(\frac{1}{4n}\right)^{\lambda/2}$$

$$\Rightarrow\; Für\ \lambda \leq 2\ Divergent$$

$$Analog:\; a_n = \left(\frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot ... \cdot \frac{(2n-3)(2n-1)}{(2n-2)^2} \cdot \frac{(2n-1)}{(2n)^2}\right)^{\lambda/2} \leq \left(\frac{2n-1}{(2n)^2}\right)^{\lambda/2} \leq \left(\frac{1}{2n}\right)^{\lambda/2}$$

$$\Rightarrow\; Für\ \lambda > 2\ Konvergent$$

g)

$$a_n := \frac{\lambda^n}{1+\lambda^n}$$

$$Fall\ 1 : \lambda \geq 1 \Rightarrow\; a_n \to 1 \Rightarrow\; Divergent$$

$$Fall\ 2 : \lambda \in [0,1) \Rightarrow\; \sqrt[n]{a_n} = \frac{\lambda}{\sqrt[n]{1+\lambda^n}} \leq \lambda < 1 \;\Rightarrow\; Konvergent$$

$$Fall\ 3 : \lambda < -1 \Rightarrow\; a_n \to 1 \Rightarrow\; Divergent$$

$$Fall\ 4 : \lambda = -1 \Rightarrow\; Nicht\ definiert$$

$$Fall\ 5 : \lambda \in (-1,0) \Rightarrow\; b_n := \frac{\lambda^n}{1+\lambda^n}\ für\ n : gerade \Rightarrow\; Konvergent (Fall\ 2)$$

$$Für\ n\ ungerade : l := |\lambda|,\; c_n := \left|\frac{\lambda^n}{1-\lambda^n}\right| = \frac{l^n}{1-l^n} > 0 \;\Rightarrow\; \frac{c_{n+1}}{c_n} = \frac{l(1-l^n)}{1-l \cdot l^n} \leq l < 1\ da\ \frac{1-l^n}{1-l \cdot l^n} < 1$$

$$\Rightarrow\; c_n\ Konvergent$$

$$Doch:\; \sum^{\infty} a_n = \sum_{n\ gerade}^{\infty} b_n - \sum_{n\ ungerade}^{\infty} c_n \;\to Konvergent$$

Methode B (Fall 5):

$$1 = \lim \sqrt[n]{1-|\lambda|} \leq \lim \sqrt[n]{1+\lambda^n} \leq \lim \sqrt[n]{1+1} = 1$$

$$\Rightarrow \lim \sqrt[n]{a_n} = \frac{\lambda}{\lim \sqrt[n]{1+\lambda^n}} = \lambda < 1$$

$$\rightarrow \textit{Konvergent}$$

h)

$$a_n := \frac{(-1)^n}{\sqrt{n}+(-1)^n}$$

$$\sum_{n=2}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n\left(\sqrt{n}-(-1)^n\right)}{n-1} = \sum_{n=1}^{\infty}\left[(-1)^n \cdot \frac{\sqrt{n}}{n-1}\right] - \sum_{n=1}^{\infty} \frac{1}{n-1}$$

$$Doch: \sum_{n=1}^{\infty} \frac{1}{n-1} > \sum_{n=1}^{\infty} \frac{1}{n} \quad Divergent$$

$$Und: \sum_{n=1}^{\infty}\left[(-1)^n \cdot \frac{\sqrt{n}}{n-1}\right] \; Konvergent\ nach\ Leipniz$$

$$\Rightarrow Divergent$$

i)

$$a_n := \lambda^{\sqrt{n}}$$

$$Fall\ 1: \lambda >= 1 \rightarrow Divergent\ (Trivial)$$

$$Fall\ 2: \lambda < 1: \sum_{n=1}^{\infty} a_n = \lambda^{\sqrt{1}} + \lambda^{\sqrt{2}} + \lambda^{\sqrt{3}} + \lambda^{\sqrt{4}} + \lambda^{\sqrt{5}} + \lambda^{\sqrt{6}} + \lambda^{\sqrt{7}} + \lambda^{\sqrt{8}} + \lambda^{\sqrt{9}} ...$$

$$\leq \left(\lambda^{\sqrt{1}} + \lambda^{\sqrt{1}} + \lambda^{\sqrt{1}}\right) + \left(\lambda^{\sqrt{4}} + \lambda^{\sqrt{4}} + \lambda^{\sqrt{4}} + \lambda^{\sqrt{4}} + \lambda^{\sqrt{4}}\right) + ... = \sum_{n=1}^{\infty} (2n+1)\lambda^{\sqrt{n^2}}$$

$$= \sum_{n=1}^{\infty} 2n\lambda^n + \sum_{n=1}^{\infty} \lambda^n = \sum_{n=1}^{\infty} 2n\lambda^n + \frac{\lambda}{1-\lambda}$$

$$b_n := 2n\lambda^n,\ \lim \sqrt[n]{b_n} = \lim \lambda \sqrt[n]{2n} = \lambda < 1 \;\Rightarrow \sum_{n=1}^{\infty} 2n\lambda^n \; Konvergent$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \; Konvergent$$

j)

$$a_n := \cos(n\pi)\sin\left(\frac{\pi}{n}\right)$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right) \ : \ Konvergent\ da\ \sin\left(\frac{\pi}{n}\right)\ Mon.Fall.\ Nullfolge\ ab\ n=2$$

Aufgabe 2

a)

$$\frac{1}{\sqrt{n}} > 0\ Mon.Fall.\ Nullfolge \rightarrow Leipniz\ Krit. \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} Konvergent \quad \square$$

b)

$$a_n := \frac{(-1)^{n+1}}{\sqrt{n}}$$

$$\left(\sum_{n=1}^{\infty} a_n\right)^2 = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} a_k \cdot a_{n-k+1}\right),\ g_k := \frac{1}{\sqrt{k}\sqrt{n-k+1}},\ c_n := \sum_{k=1}^{\infty} a_k \cdot a_{n-k+1}$$

$$\Rightarrow \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty}\left[(-1)^{n+3} \cdot \sum_{k=1}^{\infty} g_k\right]$$

$$Doch:\ \sum_{k=1}^{n} g_k \geq \sum_{k=1}^{n} \frac{1}{\sqrt{n}\sqrt{n}} = \sum_{k=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1$$

$$\Rightarrow \sum_{k=1}^{n} g_k\ Keine\ Nullfolge \Rightarrow \sum_{n=1}^{m} c_n\ Nicht\ Konvergent \quad \square$$

Aufgabe 3

$$a_n := \frac{(-1)^{n+1}}{n},\ A_N := \sum_{n=1}^{N} a_n,\ X_N := 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{N} - \ln(N) \to \gamma$$

$$A_N\ Konvergent\ (Leipniz\ Kriterium)$$

$$A_N = A_N + X_N - X_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{N} \frac{1}{n} + \ln(N) + X_N = \sum_{n=1}^{N} \frac{(-1)^{n+1} - 1}{n} + \left(\ln 2 + \ln\left[\frac{N}{2}\right]\right) + X_N$$

$$= \sum_{n=1}^{\frac{N}{2}} \frac{(-1)^{2n+1} - 1}{2n} + \sum_{n=1}^{\frac{N}{2}} \frac{(-1)^{2n-1+1} - 1}{2n-1} + \ln\left[\frac{N}{2}\right] + \ln 2 + X_N = \sum_{n=1}^{\frac{N}{2}} \frac{-2}{2n} + \ln\left[\frac{N}{2}\right] + \ln 2 + X_N$$

$$= -\sum_{n=1}^{\frac{N}{2}} \frac{1}{n} + \ln\left[\frac{N}{2}\right] + \ln 2 + X_N = -X_{\frac{N}{2}} + X_N + \ln 2$$

$$\Rightarrow\ \lim A_N = -\lim X_{\frac{N}{2}} + \lim X_N + \ln 2 = \ln 2$$

Aufgabe 4

a)

$$(|a_n| - |b_n|)^2 \geq 0 \Rightarrow a_n^2 + b_n^2 \geq 2|a_n b_n|$$

$$\Rightarrow\ 0 \leq \sum_{n=1}^{\infty} |a_n b_n| \leq \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\ \Rightarrow\ Konvergent \quad \square$$

b)

$$Aus\ (a)\ folgt: \sum_{n=1}^{\infty} a_n b_n\ auch\ Konvergent$$

$$\Rightarrow\ \sum_{n=1}^{\infty} (a_n + b_n)^2 = \sum_{n=1}^{\infty} \left[a_n^2 + b_n^2 + 2a_n b_n\right] = \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 + 2\sum_{n=1}^{\infty} a_n b_n\ auch\ konvergent \quad \square$$

c)

$$Wissen: \sum_{n=1}^{\infty} \frac{1}{n^2}\ Konvergent$$

$$Setzen\ b_n := \frac{1}{n}\ \to\ (a)\ \Rightarrow\ \sum_{n=1}^{\infty} \left|\frac{a_n}{n}\right|\ auch\ konvergent \quad \square$$