Analysis I - Serie 05
FSU Jena - WS 06/07
- Lösungen -

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Aufgabe 1

$$a_n := \frac{x_{n+1}}{x_n},\ b_n := \sqrt[n]{x_n}$$

$$\overline{\lim}\ a_n = \lim_{k\to\infty} \sup\{a_n \mid n \geq k\} = \lim_{k\to\infty} \sup\left\{ \frac{\left(\frac{1}{2}\right)^{(n+2)/2}}{\left(\frac{1}{3}\right)^{n/2}},\ \frac{\left(\frac{1}{3}\right)^{(n+1)/2}}{\left(\frac{1}{2}\right)^{(n+1)/2}} \mid n \geq k \right\}$$

$$= \lim_{k\to\infty} \sup\left\{ \frac{1}{2}\cdot\left(\frac{3}{2}\right)^{n/2},\ \sqrt{\frac{2}{3}}\cdot\left(\frac{2}{3}\right)^{n/2} \mid n \geq k \right\} = \infty$$

$$Analog:\ \underline{\lim}\ a_n = \lim_{k\to\infty} \inf\left\{ \frac{1}{2}\cdot\left(\frac{3}{2}\right)^{n/2},\ \sqrt{\frac{2}{3}}\cdot\left(\frac{2}{3}\right)^{n/2} \mid n \geq k \right\} = \lim_{k\to\infty} 0 = 0$$

$$\overline{\lim}\ b_n = \lim_{k\to\infty} \sup\left\{ \sqrt[n]{\left(\frac{1}{2}\right)^{(n+1)/2}},\ \sqrt[n]{\left(\frac{1}{3}\right)^{n/2}} \mid n \geq k \right\} = \lim_{k\to\infty} \sup\left\{ \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt[2n]{2}},\ \frac{1}{\sqrt{3}} \mid n \geq k \right\} = \frac{1}{\sqrt{2}}$$

$$Analog:\ \underline{\lim}\ b_n = \lim_{k\to\infty} \inf\left\{ \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt[2n]{2}},\ \frac{1}{\sqrt{3}} \mid n \geq k \right\} = \frac{1}{\sqrt{3}}$$

Aufgabe 2

$$a_n := \sum_{i=2}^{n} |x_i - x_{i-1}| \ \rightarrow \ Monoton\ wachsend\ \wedge\ beschränkt$$

$$\Rightarrow\ Konvergent\ \Rightarrow\ Cauchyfolge\ \Rightarrow\ \forall\ \varepsilon > 0 : \exists\ n_0 \in \mathbb{N} : \forall\ n, m > n_0 : |a_m - a_n| < \varepsilon$$

$$\Rightarrow\ |x_m - x_n| = |x_{n+1} - x_n + ... + x_m - x_{m-1}| \leq ||x_{n+1} - x_n| + ... + |x_m - x_{m-1}|| = |a_m - a_n| \leq \varepsilon$$

$$\Rightarrow\ x_n\ Cauchyfolge\ \Rightarrow\ x_n\ konvergent$$

$Beispiel:$

$$x_n := \left\{ \begin{array}{ll} -\frac{1}{n} & : n\ gerade \\ \\ \frac{1}{n+1} & : n\ ungerade \end{array} \right\} \rightarrow\ Konvergent\ (\rightarrow 0)$$

$$Doch: \sum_{i=2}^{n} |x_i - x_{i-1}| = \left|-\frac{1}{2} - \frac{1}{2}\right| + \left|\frac{1}{4} + \frac{1}{2}\right| + ... + \left|-\frac{1}{n} - \frac{1}{n}\right| = 1 + \frac{3}{4} + \frac{1}{2} + ... + \frac{2}{n}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \left(\frac{3}{4} + ...\right) \rightarrow \infty\ da\ \frac{1}{n}\ divergent$$

Aufgabe 3

$$Es\ gilt: e = \lim\left(1 + \frac{1}{n}\right)^n = \lim\left(1 + \frac{1}{n}\right)^{n+1}\ \wedge\ \left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$$

$$\Rightarrow\ \frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$$

$$\Rightarrow\ x_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln(n)\ >\ \ln\left(1 + \frac{1}{1}\right) + \ln\left(1 + \frac{1}{2}\right) + ... + \ln\left(1 + \frac{1}{n}\right) - \ln(n)$$

$$= \ln\left(\frac{2 \cdot 3 \cdot 4 \cdot ... \cdot (n+1)}{1 \cdot 2 \cdot ... \cdot n}\right) - \ln(n) = \ln(n+1) - \ln(n) > \ln(n) - \ln(n) = 0$$

$$x_{n+1} - x_n = \frac{1}{n+1} + \ln(n) - \ln(n+1) = \frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right)\ <\ \ln\left(1 + \frac{1}{n}\right) + \ln\left(\frac{n}{n+1}\right) = 0$$

$$\Rightarrow\ x_n\ monoton\ fallend\ \wedge\ und\ beschränkt$$

$$\Rightarrow\ x_n\ konvergent\ \ \square$$

Aufgabe 4

a)

$$a_n := \frac{2n-1}{2^n},\ \lim \sqrt[n]{a_n} = \lim \frac{1}{2} \cdot \sqrt[n]{2n-1} \leq \lim \frac{1}{2} \cdot \sqrt[n]{2n} = \lim \frac{1}{2} \cdot \sqrt[n]{2} \cdot \sqrt[n]{n} = \frac{1}{2} \cdot \lim \frac{n+1}{n} = \frac{1}{2} < 1$$

$$\Rightarrow \sum_{i=1}^{n} a_n \ konvergent,\ S := \sum_{i=1}^{\infty} a_n$$

$$S - \frac{S}{2} = \lim_{k\to\infty} \left(\sum_{n=1}^{k} \frac{2n-1}{2^n} - \sum_{n=1}^{k} \frac{2n-1}{2^n \cdot 2} \right) = \frac{1}{2} + \lim_{k\to\infty} \left(\sum_{n=2}^{k} \frac{2n-1}{2^n} - \sum_{n=1}^{k-1} \frac{2n-1}{2^n \cdot 2} - \frac{2k-1}{2^k \cdot 2} \right)$$

$$= \frac{1}{2} + \lim_{k\to\infty} \left(\sum_{n=1}^{k-1} \left[\frac{2(n+1)-1}{2^n \cdot 2} - \frac{2n-1}{2^n \cdot 2} \right] \right) + \lim_{k\to\infty} \frac{2k-1}{2^k \cdot 2} = \frac{1}{2} + \lim_{k\to\infty} \sum_{n=1}^{k-1} \frac{1}{2^n} = \frac{1}{2} + 1 = \frac{3}{2} \ \Rightarrow \ S = 3$$

b)

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} \left(\sum_{n=1}^{k} \frac{1}{n(n+1)} - \sum_{n=1}^{k} \frac{1}{n} + \sum_{n=1}^{k} \frac{1}{n} \right) = \lim_{k\to\infty} \left(- \sum_{n=1}^{k} \frac{1}{n+1} + \sum_{n=1}^{k} \frac{1}{n} \right)$$

$$= \lim_{k\to\infty} \left(- \sum_{n=2}^{k+1} \frac{1}{n} + \sum_{n=1}^{k} \frac{1}{n} \right) = \lim_{k\to\infty} \left(1 - \frac{1}{k+1} - \sum_{n=1}^{k} \frac{1}{n} + \sum_{n=1}^{k} \frac{1}{n} \right) = \lim_{k\to\infty} \left(1 - \frac{1}{k+1} \right) = 1$$

c)

$$F\ddot{u}r\ q \geq 1 \to \sum_{n=1}^{\infty} q^n = \infty$$

$$Sonst: \ \sum_{n=1}^{\infty} q^n = \lim_{k\to\infty} \left(\frac{(q-1)}{(q-1)} \cdot \sum_{n=1}^{k} q^n \right) = \frac{1}{q-1} \cdot \lim_{k\to\infty} \left(\sum_{n=1}^{k} \left[q^{n+1} - q^n \right] \right) = \frac{1}{q-1} \cdot \lim_{k\to\infty} \left(q^{k+1} - q \right) = \frac{q}{1-q}$$