Analysis I - Serie 03
FSU Jena - WS 06/07
- Lösungen -

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Aufgabe 1

$$c := max(a,b) = \frac{a+b+|a-b|}{2}$$

$$\forall\ \varepsilon > 0 : \exists n_0 \in \mathbb{N} : \forall\ n > n_0 : |a - a_n| < \frac{\varepsilon}{2}\ \wedge\ |b - b_n| < \frac{\varepsilon}{2}$$

$$\Rightarrow |c - c_n| = \left| \frac{a+b+|a-b| - a_n - b_n - |a_n - b_n|}{2} \right| \leq \left| \frac{a-a_n}{2} \right| + \left| \frac{b-b_n}{2} \right| + \left| \frac{|a-b| - |a_n - b_n|}{2} \right|$$

$$\leq \frac{\varepsilon}{2} + \left| \frac{a - b - a_n + b_n}{2} \right| \leq \frac{\varepsilon}{2} + \left| \frac{a-a_n}{2} \right| + \left| \frac{b_n - b}{2} \right| \leq \varepsilon$$

$$\Rightarrow c = \lim_{n\to\infty} c_n \quad \square$$

Aufgabe 2

$$c_n := \frac{a_1 + a_2 + ... + a_n}{n}$$

$$\lim a_n = a \Rightarrow \forall\ \varepsilon > 0 : \exists n_0 \in \mathbb{N} : \forall\ n > n_0 : |a_n - a| < \frac{\varepsilon}{2}$$

$$\to w\ddot{a}hlen : m_0 > n_0 \in \mathbb{N} : \frac{\sum_{i=1}^{n_0} |a_i - a|}{m_0} < \frac{\varepsilon}{2}$$

$$\Rightarrow\ \forall\ n > m_0 : |c_n - a| = \left| \frac{a_1 + a_2 + ... + a_n - n \cdot a}{n} \right| \leq \frac{|a_1 - a| + |a_2 - a| + ... + |a_{n_0} - a| + |a_{n_0+1} - a| + ... + |a_n - a|}{n}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon \cdot (n - n_0)}{2n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \Rightarrow \quad \lim_{n\to\infty} c_n = a \quad \square$$

Aufgabe 3

a)

$$\lim \left(\frac{1}{n} \cdot \sum_{i=1}^{n} \frac{1}{a_i} \right) = \frac{1}{a} \ (für \ a \neq 0) \ \wedge \ \lim \left(\frac{\sum_{i=1}^{n} a_i}{n} \right) = a \ (Aufgabe \ 2)$$

$$Ausserdem \ gilt: \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \leq \sqrt[n]{\prod_{i=1}^{n} a_i} \leq \frac{\sum_{i=1}^{n} a_i}{n}$$

$$\Rightarrow a = \lim \frac{1}{\frac{1}{n} \cdot \sum_{i=1}^{n} \frac{1}{a_i}} \leq \lim \sqrt[n]{\prod_{i=1}^{n} a_i} \leq \lim \left(\frac{\sum_{i=1}^{n} a_i}{n} \right) = a$$

$$Für \ a = 0 : 0 \leq \lim \sqrt[n]{\prod_{i=1}^{n} a_i} \leq 0$$

$$\Rightarrow \lim \sqrt[n]{\prod_{i=1}^{n} a_i} = a \quad \square$$

b)

$$Def.: a_0 = 1$$

$$\lim \frac{a_n}{a_{n-1}} = \lim \sqrt[n]{\frac{a_1}{a_0} \cdot \frac{a_2}{a_1} \cdot ... \cdot \frac{a_n}{a_{n-1}}} = \lim \sqrt[n]{a_n} \quad \square$$

Aufgabe 4

a)

$$\lim_{n \to \infty} a_n = \lim \frac{n+2-(n-1)}{\sqrt{n+2}+\sqrt{n-1}} = \lim \frac{3}{\sqrt{n+2}+\sqrt{n-1}} = 0$$

b)

$$\lim_{n \to \infty} a_n = \lim \frac{a^n}{1+a^n} = \lim \frac{a^n}{a^n \left(\frac{1}{a^n} + 1 \right)} = \frac{1}{\lim \frac{1}{a^n} + 1}$$

$$\Rightarrow \left(a = 1 \ \Rightarrow \ \lim a_n = \frac{1}{2} \right) \wedge (|a| < 1 \ \Rightarrow \ \lim a_n = 0) \wedge (|a| > 1 \ \Rightarrow \ \lim a_n = 1) \wedge (a = -1 \ \Rightarrow \ \neg \exists \lim a_n)$$

c)

$$7 \leq a_n = 7 \cdot \sqrt[n]{1 + \left(\frac{3}{7} \right)^n + \left(\frac{5}{7} \right)^n} \leq 7 \sqrt[n]{3}$$

$$\Rightarrow \ 7 \leq \lim_{n \to \infty} a_n \leq \lim 7 \sqrt[n]{3} = 7 \ \Rightarrow \ \lim a_n = 7$$

d)

$$b := |a|,\ c := \lceil b \rceil \geq b \ \Rightarrow \frac{b}{c+1} < 1,\ M := \frac{b \cdot b \cdot ... \cdot b}{1 \cdot 2 \cdot ... \cdot c}$$

$$0 \leq \lim_{n\to\infty} |a_n| = \lim \frac{b^n}{n!} = \lim \frac{b \cdot \cdot b \cdot ... \cdot b}{1 \cdot 2 \cdot ... \cdot c \cdot ... \cdot n} = \lim M \cdot \frac{b}{c+1} \cdot \frac{b}{c+2} \cdots \frac{b}{n} \leq \lim M \cdot \left(\frac{b}{c+1}\right)^{n-c} = 0 \ \ \textit{da M beschränkt}$$

$$\Rightarrow \lim a_n = 0$$

e)

$$\lim_{n\to\infty} a_n = \lim \sqrt[n]{\binom{2n}{n}} = \lim \frac{\binom{2n+2}{n+1}}{\binom{2n}{n}} = \lim \frac{\frac{(2n+2)!}{(n+1)!\cdot(n+1)!}}{\frac{(2n)!}{n!\cdot n!}} = \lim \frac{(2n+1)(2n+2)}{(n+1)(n+1)}$$

$$= \lim \frac{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)} = \frac{2 \cdot 2}{1 \cdot 1} = 4$$

f)

$$\lim_{n\to\infty} a_n = \lim \frac{n(n-1)}{2n^2} = \lim \frac{n-1}{2n} = \lim \frac{1-\frac{1}{n}}{2} = \frac{1-\lim\frac{1}{n}}{2} = \frac{1}{2}$$